Strongly Interacting Fermi Gases

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Croucher Summer Course: Ultracold Atom Physics

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Plan:

Lecture I:

Introduction to p-wave system and its comparison with its s-wave counterpart;
Low-energy scattering and universal relations;
Normal state and Fermi liquid properties;
Three-body recombination.

Lecture II:

Bulk viscosity and effects of effective range; connection to SO(2,1) symmetry;
Viscosity sum rule and high frequency tails;
Viscosity in 1D p-wave Fermi system;
Emergent s-wave scattering in 1D p-wave Fermi system.

Low-energy scattering

Consider low-energy scattering via a short-range potential ($\sim r_0$). The effects of the potential can be described by a few scattering parameters via effective range expansion.

In the s-wave case, $k \cot \delta_s(k) = -1/a_s - r_e k^2 + \cdots$

$$f_s(k) = \frac{1}{-1/a_s - r_e k^2 - ik}$$

The s-wave effective range is sub-leading to *ik*; its effects usually is perturbative.

In the p-wave case, $k^3 \cot \delta_p(k) = -1/v - r_1 k^2 + \cdots$

$$f_p(k) = \frac{k^2 \cos\chi}{-1/v - r_1 k^2 - ik^3}$$

The p-wave effective range is more important than the unitary term ik^3.

Close to resonance ($v \to \infty$), causality bound shows that $r_1 > 0$ and cannot be taken to be zero, so unitary limit for p-wave scattering cannot be achieved [Hammer and Lee, 2010; Bo Gao 2011, Yu, Thywissen and Zhang, 2015].

Proof of $r_1 > 0$

Let us sketch an explicit proof of $r_1>0$. Consider the radial Schrödinger equation with $\ell = 1$:

$$\left[-\frac{d^2}{dr^2} + \frac{2}{r^2} + \frac{m}{\hbar^2}v(r)\right]\chi(r) \equiv \hat{L}\chi(r) = k^2\chi(r)$$

Here $k^2 = mE/\hbar^2$. Now the wave function $\chi(r)$ can be expanded in energy as

$$\chi(r) = u_0 + k^2 u_1(r) + \cdots$$

And the Schrödinger equation can be written as:

$$\hat{L}u_0 = 0$$

$$\hat{L}u_1 = u_0$$

Let us define the Green function $G(r, r')$ as: $\hat{L}G(r, r') = \delta(r - r')$. Now if we take $u_0(r)$ and $v_0(r)$ to be the regular and irregular solution of the homogeneous equation at $r = 0$, then

$$u_1(r) = \int_0^r G(r, r')u_0(r')dr'$$

$$G(r, r') = \frac{1}{W}\Theta(r - r')u_0(r)v_0(r') - v_0(r)u_0(r'))$$

W is the Wronskian (constant).

Proof of $r_1 > 0$

Using this form of Green function, one can write down the wave function $\chi(r)$ as

$$\chi(r) = u_0(r) + \frac{k^2}{W}\int_0^r dr'[u_0(r)v_0(r') - v_0(r)u_0(r')]u_0(r')$$

Let us consider a point r_c for which $r_0 \ll r_c \ll d$, where d is the average inter-particle distance, then the wave function can be written in terms of asymptotic wave functions

$$\chi(r) = r(\cot\delta_p(k) j_1(kr) - y_1(kr))$$

Here $\delta_p(k)$ is the p-wave scattering phase shift

$$k^3 \cot\delta_p(k) = -\frac{1}{v} - r_1 k^2 + \cdots$$

By equating the logarithm derivative of $\chi(r)$ at r_c, one finds the equation

$$r_1 = \int_0^{r_c} u_0(r')^2 dr' + \frac{r_c^2}{3v} + \frac{1}{r_c} - \frac{r_c^5}{45v^2}$$

This is always large than zero close to resonance when $v \to \pm\infty$.

Relation of (v, r_1) to inter-atomic potential

We can establish the relation between the short-range potential and the low-energy scattering parameters, (v, r_1). Consider two nearly identical potential $v_\pm(r) = v(r) \pm \delta v(r)/2$, the SEs are:

$$\left(-\frac{\hbar^2}{m}\frac{d^2}{dr^2} + v_\pm(r) + \frac{2\hbar^2}{mr^2}\right)\chi_\pm(r) = \frac{\hbar^2 k_\pm^2}{m}\chi_\pm(r)$$

The wave functions in the range $r \sim r_c$ take the form:

$$\chi_\pm(r) = \left(\frac{1}{r} - \frac{r^2}{3v_\pm}\right) + k_\pm^2\left(\frac{r}{2} - \frac{r^2}{3}r_1 + \frac{r^4}{30v_\pm}\right) + \cdots$$

By the usual trick, and expand to $k_\pm^2$ and $\delta v(r)$, then we have

$$-\delta(1/v) - (k_+^2 + k_-^2)/2\delta r_1 = \frac{m}{\hbar^2}\int_0^\infty dr\delta v(r)\chi_+(r)\chi_-(r)$$

This allows for the following identities:

$$\delta v^{-1} = -\frac{m}{\hbar^2}\int_0^\infty dr\,\delta v(r)|u_0(r)|^2,$$

$$\delta r_1 = -\frac{2m}{\hbar^2}\int_0^\infty dr\,\delta v(r)u_0(r)u_1(r),$$

Universal relations

Due to the separation of length scales $d \gg r_0$, the short-range correlation can be approximated by that of the two-body correlation. Formally, this means that the two-body density matrix can be written as

$$\rho_2(\mathbf{r}_1, \mathbf{r}_2) = \sum_\alpha n_\alpha \phi^*_\alpha(\mathbf{r}_1, \mathbf{r}_2)\phi_\alpha(\mathbf{r}_1, \mathbf{r}_2).$$

With $\phi_\alpha(\mathbf{r})$ etc being the eigenfunction and n_α eigenvalues. Importantly, $\phi_\alpha(\mathbf{r})$ must have the same form as two-body physics in the range $r \leq r_0$. That is, in the short-range, we can write

$$\phi_{\alpha=(\mathbf{P},j,\ell,m)}(\mathbf{r}_1, \mathbf{r}_2) = \frac{1}{\sqrt{\Omega} r} \exp(i\mathbf{P}\cdot\mathbf{R})\varphi_{j\ell}(r)Y_{\ell m}(\hat{r}),$$

$$\varphi_{j\ell=1}(r) = \int_0^\infty dk a_{jk}\chi_k(r) + a_{j\kappa}\chi_\kappa(r),$$

These relations allows for a straightforward derivation of several universal relations. For example, let us consider the interaction energy in a many-body system. Recall that the potential $v(r)$ is a short-range function. It can be written as

$$\langle v \rangle = \frac{1}{2}\int d\mathbf{r}_1 d\mathbf{r}_2 v(|\mathbf{r}_1 - \mathbf{r}_2|)\rho_2(\mathbf{r}_1 - \mathbf{r}_2)$$

$$\langle v \rangle = \frac{1}{2}\sum_m \left[C_v^{(m)} \int dr v(r)|u_0(r)|^2 + C_R^{(m)} \int dr v(r)u_0(r)u_1(r)\right]$$

Universal relations

We have defined the so-called contact parameters

$$C_v^{(m)} = \sum_{\mathbf{P},j} n_{\mathbf{P},j,m} (\int dk a_{jk})^2,$$

$$C_R^{(m)} = \frac{1}{2} \sum_{\mathbf{P},j} n_{\mathbf{P},j,m} \int dk \int dk' a_{jk} a_{jk'} (k^2 + k'^2).$$

We note that clearly $C_v^{(m)} > 0$. Consider a small change of interaction potential $\delta v(r)$, or more explicitly $(dv/d\lambda)d\lambda$, then by using Hellmann-Feynman theorem,

$$\frac{dE}{d\lambda} = \left\langle \frac{dv}{d\lambda} \right\rangle$$

This again can be expressed in terms of

$$\left\langle \frac{dv(r)}{d\lambda} \right\rangle = \frac{1}{2} \sum_m \left[C_v^{(m)} \int dr \frac{dv(r)}{d\lambda} |u_0(r)|^2 + C_R^{(m)} \int dr \frac{dv(r)}{d\lambda} u_0(r) u_1(r) \right]$$

Using the relations between δv^{-1}, δr_1 and $\delta v(r)$, one finds the adiabatic relations.

$$\delta v^{-1} = -\frac{m}{\hbar^2} \int_0^\infty dr\, \delta v(r) |u_0(r)|^2,$$

$$\delta r_1 = -\frac{2m}{\hbar^2} \int_0^\infty dr\, \delta v(r) u_0(r) u_1(r),$$

Universal relations

Adiabatic relations:

$$\left.\frac{dE}{dv^{-1}}\right|_{r_1} = -\frac{\hbar^2}{m}\sum C_v^{(m)} \qquad \left.\frac{dE}{dr_1}\right|_{r_1} = -\frac{\hbar^2}{m}\sum C_R^{(m)}$$

Simple example of two-body bound state (say for $m = 0$ component, pole of $f_p(k)$)

$$E = -\frac{\hbar^2}{mvr_1}$$

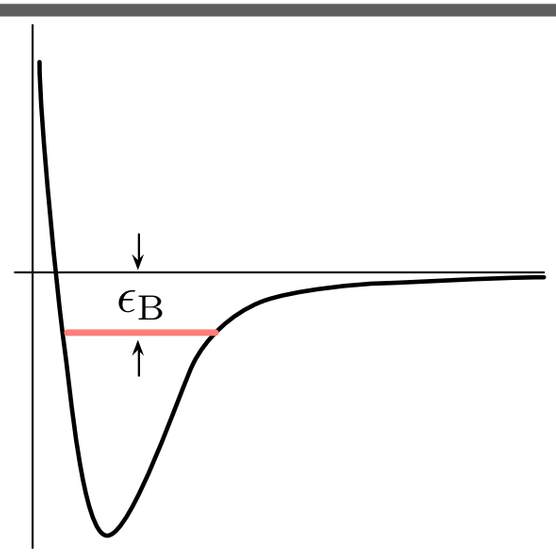
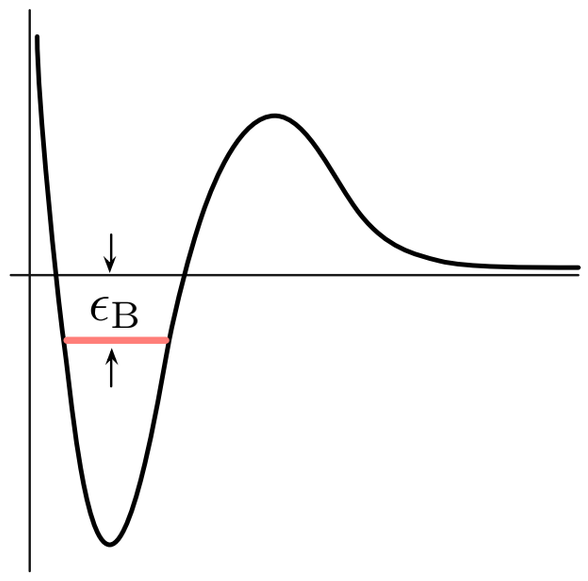
This gives the two-body contact parameters

$$C_v = \frac{1}{r_1} \qquad C_R = -\frac{1}{vr_1^2}$$

This is consistent with the direct analysis of the two-body wave function $\psi(r) = \sqrt{1/r_1}(1/r^2 + \kappa/r)\exp(-\kappa r)Y_{\ell=1m}(\hat{r})$.

As long as one can describe the p-wave system by two parameters (v, r_1), those universal relations are quite independent of the exact model for the microscopic interaction. [Yoshida and Ueda, PRL 2015, Yu, Thywissen and Zhang, PRL 2015, He, Zhang, Chan and Zhou, PRL 2016.]

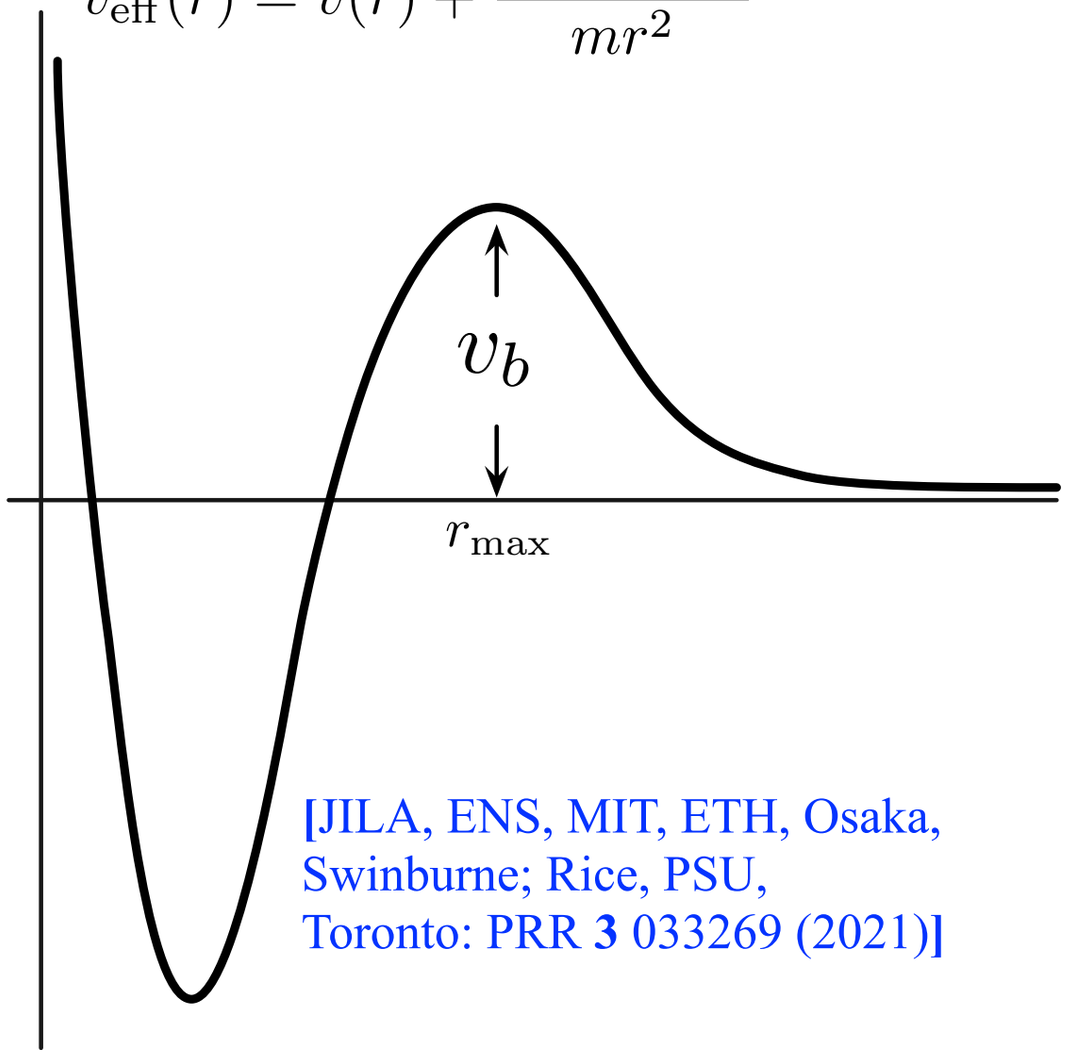
Two-Body Bound State

	RADIAL POTENTIAL	WAVE FUNC.	ENERGY	REMARK
S-WAVE		$\sim \frac{1}{r}\exp\left(-\frac{r}{a_s}\right)$	$\frac{\hbar^2}{ma_s^2}$	exists for $a_s > 0$
P-WAVE		$\sim \sqrt{1/r_1}\left(\frac{1}{r^2}+\frac{\kappa}{r}\right)$ $\times \exp(-\kappa r)\, Y_{1m}(\hat{r})$	$\frac{\hbar^2}{mvr_1}$	$r_1 > 0$ true b.s. $v > 0$ quasi-b.s $v < 0$

Inter-atomic potential

In ultra cold atomic systems, the inter-atomic potential $v(r)$ generally consists of a short-range repulsive core and long range attractive van der Waals tail.

$$v_{\text{eff}}(r) = v(r) + \frac{\hbar^2 \ell(\ell+1)}{mr^2}$$



[JILA, ENS, MIT, ETH, Osaka, Swinburne; Rice, PSU, Toronto: PRR **3** 033269 (2021)]

For cold atoms in their ground electronic state, $v(r)$ usually has a long tail given by $-C_6/r^6$, and the form of $v_{\text{eff}}(r)$ becomes (» hardcore)

$$v_{\text{eff}}(r) = -\frac{C_6}{r^6} + \frac{\hbar^2 \ell(\ell+1)}{mr^2}$$

	C_6	(VdW) r_0	v_b
6Li	1389	$64.7a_B$	123.2 MHz
40K	3897	$129a_B$	5.0 MHz

With VdW, the p-wave effective range expansion is generally not correct (Zhang, Pascal and Ueda, 2010; Gao, 2011).

Inter-atomic potential: General consideration

One needs to simplify the actual interaction while retaining the same low-energy scattering properties.

$$\epsilon_{\mathbf{k}}\psi_{\mathbf{k}} + \sum_{\mathbf{k}'} v_{\mathbf{k}\mathbf{k}'}\psi_{\mathbf{k}'} = E\psi_{\mathbf{k}}$$

The inter-atomic potential can be decomposed in angular momentum channels

$$v_{\mathbf{k}\mathbf{k}'} = \sum_{\ell}(2\ell+1)v_{kk'}^{(\ell)}p_{\ell}(\cos\theta)$$

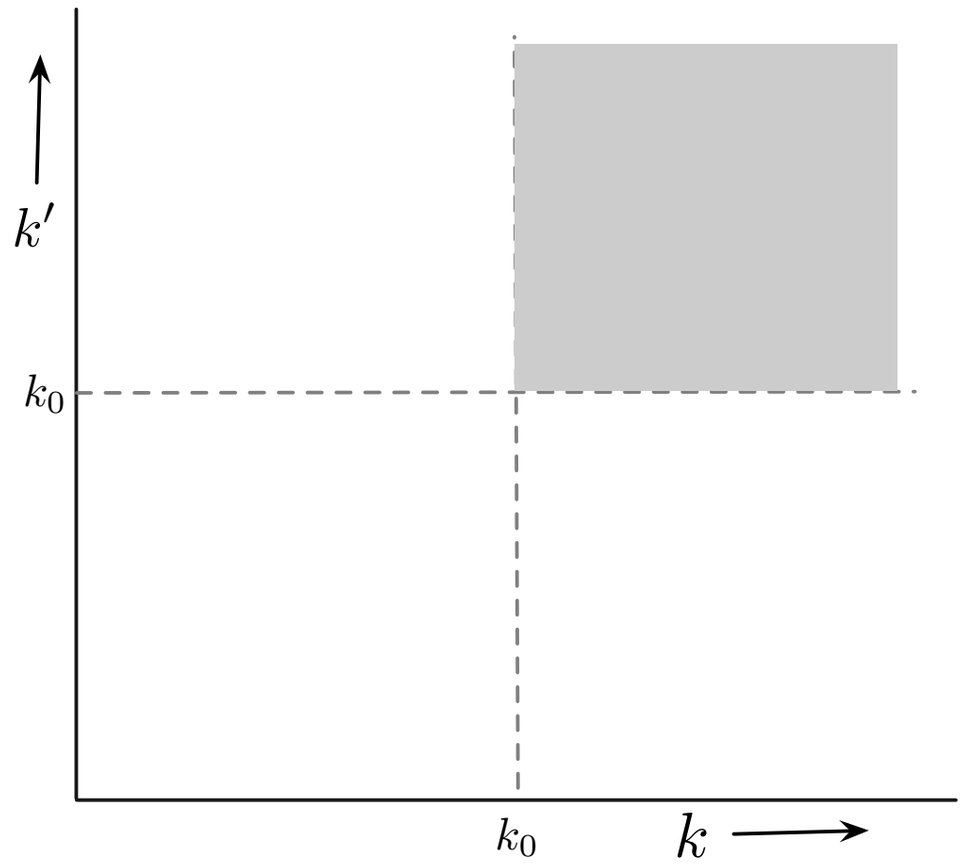
Focus only on the p-wave channel, we have

$$v_{kk'}^{\ell=1} = \int dr j_1(kr)v(r)j_1(k'r)r^2dr$$

Its structure as a function of k, k' can be determined when either of them is small.

When $k \ll 1/r_0$, $\quad j_1(kr) \sim kr$ $v_{kk'}^{\ell=1} = k\bar{g}(k')$

$$v_{kk'}^{\ell=1} = kk'g(k)g(k')$$



Model Interaction

Thus, one possible choice of the effective potential is given by choosing

$$g(k) = c_0 + c_1 k^2$$

Neglecting the fourth order term in momentum,

$$v_{kk'}^{\ell=1} = kk'[g_1 + g_2(k^2 + k'^2)]$$

This leads to a different set of renormalization (perturbative) conditions:

$$g_1 = \frac{12\pi v}{m} + \frac{24v^2}{m}\int_0^\Lambda dq q^2$$

$$g_2 = -\frac{3\pi v^2 r_1}{m} + \frac{12v^2}{m}\int_0^\Lambda dq$$

For potential in the weak coupling limit (g_1, g_2 small), the expansion amounts to keeping terms of order v, v^2 and $v^2 r_1$. In the Fermi liquid calculation, we will retain up to v^2 and $v^2 r_1$ (assuming r_1 to be a constant).

S-wave vs. P-wave

Renormalization condition is fixed by calculating the on-shell scattering amplitude. In s-wave only up to energy independent term:

$$f_s(k) = \frac{1}{-a_s^{-1} - ik} \qquad g = \frac{4\pi a_s}{m} - \frac{a_s^2}{2\pi^2 m}\int dq$$

This guarantees the finiteness of off-shell scattering amplitude also (Fetter & Walecka).

In the p-wave case, renormalisation needs to be done up to the effective range order

$$f_p(k) = \frac{k^2 \cos\chi}{-1/v - r_1 k^2 - ik^3}$$

Consequence:

(1) In the p-wave case, the renormalisation condition in general only guarantees the finiteness of on-shell scattering amplitude;

(2) In the many-body calculation beyond mean field, off-shell scattering amplitudes are also needed → consistent calculation of off-shell properties then requires extra parameters other than v and r_1 (quasi-particle residue!)

S-wave vs. P-wave

To make concrete the last statement, we look at (half) off-shell scattering amplitude

$$f_p(k', k) = \int_0^\infty J_1(k'r)V(r)\psi_p(k, r)dr$$

If $k = k'$, then we have the scattering amplitude given before. To analyse the structure of $f_p(k', k)$ when both k' and k are small, we note

$$J_1(k'r) \propto k'r(1 - k'^2r^2/8 + \cdots)$$

$$\psi_p(k, r) = k(\psi_p^{(0)}(r) + k^2\psi_p^{(1)}(r) + \cdots)$$

Substitute in $f_p(k', k)$, we find

$$f_p(k', k) = kk' \left[\int_0^\infty r\psi_p^{(0)}(r)V(r)dr + k^2 \int_0^\infty r\psi_p^{(1)}V(r)dr - \frac{k'^2}{8} \int_0^\infty r^3\psi_p^{(0)}V(r)dr\right]$$

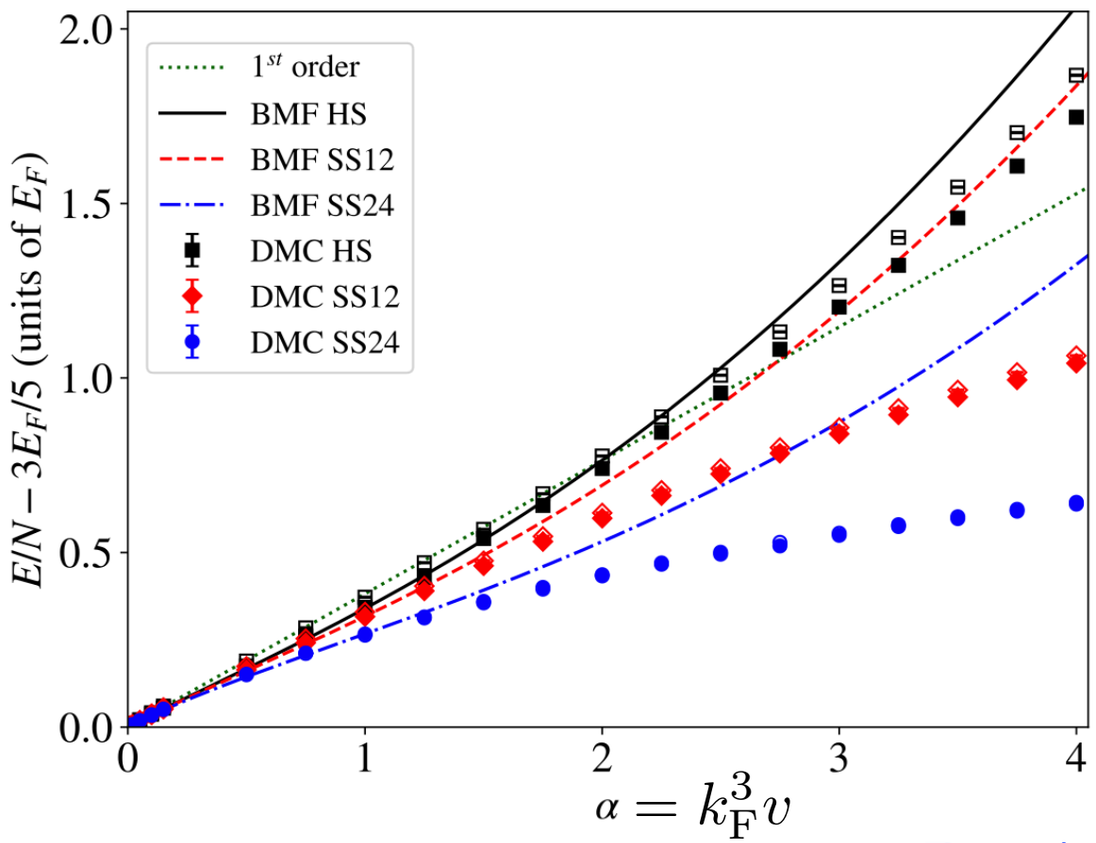
The first term corresponds to the scattering volume v, but later two terms cannot be expressed in terms of on-shell scattering parameter r_1 alone. Needs more microscopic parameters! One consequence: quasi-particle residue cannot be expressed in terms of v and r_1.

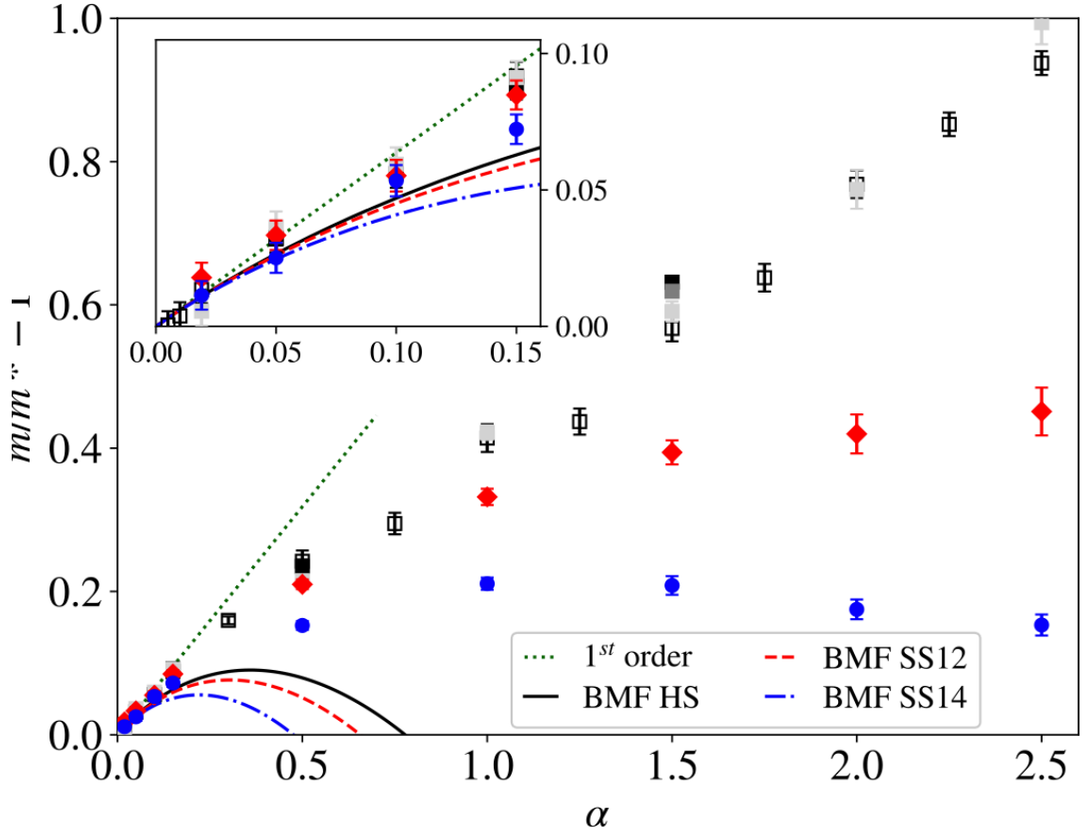
Fermi liquid properties

Using the (g_1, g_2) form of the interaction potential, one can compute the Landau Fermi liquid properties of a single component Fermi gas [Ding and Zhang, PRL 2019]

Ground state energy: $$\frac{E}{N} = \epsilon_{\mathrm{F}} \left(\frac{3}{5} + \frac{6}{5\pi} k_{\mathrm{F}}^3 v - \frac{9}{35\pi} \frac{(k_{\mathrm{F}}^3 v)^2}{k_{\mathrm{F}} R} + \frac{108}{11\pi^2} \frac{1033 - 156 \ln 2}{5672} (k_{\mathrm{F}}^3 v)^2 \right)$$

Effective mass: $$\frac{m}{m*} = 1 + \frac{2k_{\mathrm{F}}^3 v}{\pi} - \frac{1}{\pi} \frac{(k_{\mathrm{F}}^3 v)^2}{k_{\mathrm{F}} R} + \frac{18}{\pi^2} \frac{4(313 - 426 \ln 2)}{2835} (k_{\mathrm{F}}^3 v)^2$$





Bertaina,Tarallo and Pilati PRA 107, 053305 (2023)

Fermi liquid properties

Landau function: $$f(\chi) = \frac{\pi}{mk_F}\Bigg\{12(k_{\rm F}^3 v)\,(1-\cos\chi) - 6\frac{r_1}{k_{\rm F}}(k_{\rm F}^3 v)^2\,(1-\cos\chi)^2 + \frac{\pi(k_{\rm F}^3 v)^2}{16}\csc\frac{\chi}{2}\Bigg[(1428\sin\frac{\chi}{2} - 674\sin\frac{3\chi}{2} + 330\sin\frac{5\chi}{2}) + \frac{3}{2}\ln\frac{1+\sin\frac{\chi}{2}}{1-\sin\frac{\chi}{2}}(153\cos\chi - 30\cos 2\chi + 55\cos 3\chi - 18)\Bigg]\Bigg\}$$

Note that the same logarithm singularity occurs when $\chi = \pi$. The dimensionless Landau parameter $F(\cos\chi) = \nu(0) f(\cos\chi) = \sum_\ell F_\ell P_\ell(\cos\chi)$

$$F_0 = \frac{1}{\pi}\left[6(k_{\rm F}^3 v) + 4\left(\frac{r_1}{k_{\rm F}} - \frac{12\ln 2 + 11}{7\pi}\right)(k_{\rm F}^3 v)^2\right]$$

$$F_1 = \frac{1}{\pi}\left[-6(k_{\rm F}^3 v) + \left(\frac{6r_1}{k_{\rm F}} + \frac{4(852\ln 2 - 311)}{105\pi}\right)(k_{\rm F}^3 v)^2\right]$$

$$F_2 = \frac{1}{\pi}\left(-2\frac{r_1}{k_{\rm F}} + \frac{8(66\ln 2 - 13)}{21\pi}\right)(k_{\rm F}^3 v)^2$$

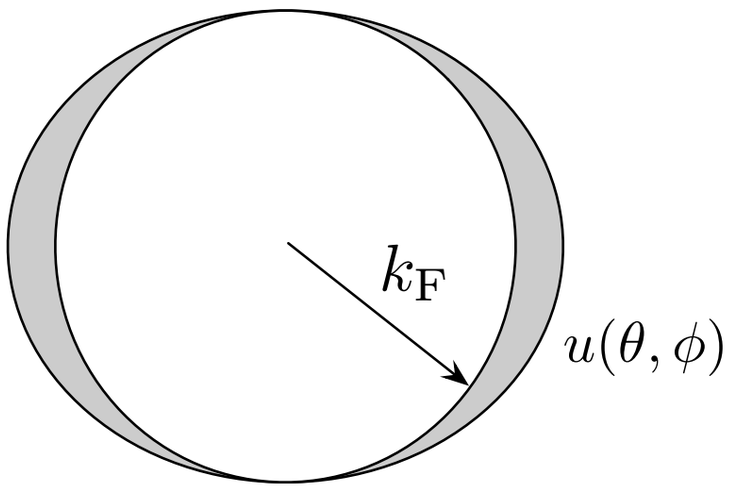
Exact Fermi liquid relations holds: $$\frac{1}{m^*} = \frac{1}{m} - \frac{k_{\rm F}}{(2\pi)^3}\int f(\chi)\cos\chi d\Omega$$

$$c_1^2 = k_{\rm F}^2(1+F_0)/(3mm^*)$$

In the limit when r_1 vanishes, the contribution from effective range vanishes.

Fermi liquid properties

Collective excitations are quite different from *s*-wave case.



$$(\cos\theta-\lambda)u(\theta,\phi)+\frac{\cos\theta}{4\pi}\int d\Omega' F(\chi)u(\theta',\phi')=0$$

To first order in v:

$$F(\chi)=F_0+F_1\cos\chi$$

To have undamped collective mode: **λ**>1.

1. Zero sound (m=0): $u(\theta,\phi)=u(\theta)$ $I_n(\lambda)=\int_{-1}^{1}dx x^n/(x-\lambda)$

$$\left(1+\frac{1}{2}F_0I_1(\lambda)\right)c_0+\frac{1}{6}F_1I_2(\lambda)c_1=0$$

$$\frac{1}{2}F_0I_2(\lambda)c_0+\left(\frac{1}{3}+\frac{1}{6}F_1I_3(\lambda)\right)c_1=0$$

No solution for **λ**>1.

Fermi liquid properties

Zero sound to v^2. Define $\nu(\theta) = (\cos\theta - \lambda)u(\theta)$

$$\nu(\theta) + \frac{\cos\theta}{4\pi}\int d\Omega' F(\chi)\frac{\nu(\theta')}{\cos\theta' - \lambda} = 0$$

For weak coupling $F(\chi) \approx 0$, it is easy to see that $\lambda \to 1$, and the major contribution to the integral arises from $\theta' \approx 0$. $\nu(\theta)$ is nonzero only in small region close to $\theta \approx 0$.

$$\lambda = 1 + 2\exp\left[-\frac{1}{24(k_{\mathrm{F}}^3 v)^2}\right]$$

Namely, zero sound only exists due to many-body corrections, similar to 2D dipolar Fermi gas (Lu and Shlyapnikov, PRA, 2012).

2. Transverse sound (m=1): $u(\theta,\phi) = u(\theta)\exp(i\phi)$

requires $F_1 > 6$, thus no undamped transverse sound (not surprising!).

Normal state properties

Analytic calculation at low-temperature cannot be extended to strongly interacting regime, but it can be done at high temperature approximately using Nozieres-Schmitt-Rink scheme.

Two-channel model:

$$H_0 = \sum_{\mathbf{k}} \epsilon_{\mathbf{k}} a^\dagger_{\mathbf{k}} a_{\mathbf{k}} + \sum_{m,\mathbf{q}} (\epsilon_{\mathbf{q}}/2 - \nu_m) b^\dagger_{m,\mathbf{q}} b_{m,\mathbf{q}}$$

$$V = \sum_{m,\mathbf{k},\mathbf{q}} \frac{g_m}{\sqrt{2\Omega}} k Y_{1m} a^\dagger_{\mathbf{q}/2-\mathbf{k}} a^\dagger_{\mathbf{q}/2+\mathbf{k}} b_{m,\mathbf{q}} + \text{H.c.}$$

Calculating the two-body scattering T-matrix, one can find the renormalisation conditions

$$v_m^{-1} = \frac{4\pi}{g_m^2 m} \nu_m + \frac{4\pi}{\Omega} \sum_{\mathbf{k}} 1$$

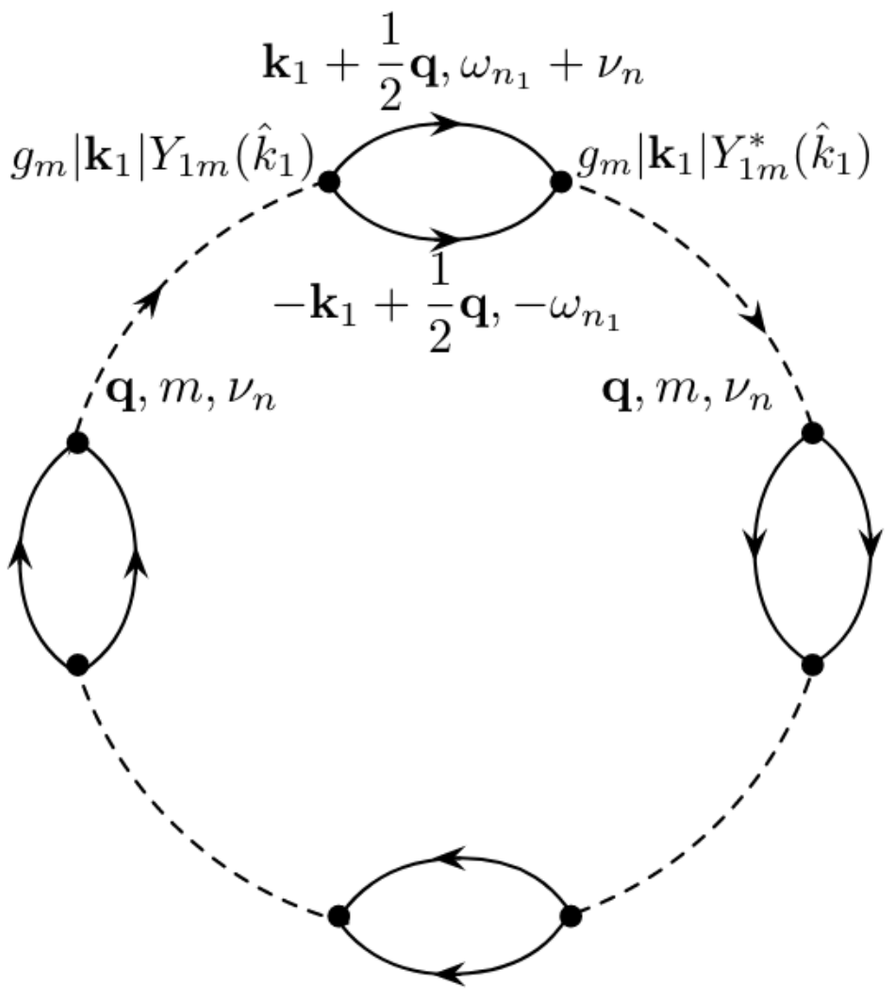
$$r_m = \frac{4\pi}{g_m^2 m^2} + \frac{2\pi}{m\Omega} \sum_{\mathbf{k}} \frac{1}{\epsilon_{\mathbf{k}}}$$

Note again while v_m can have both signs, r_m is always positive.

Normal state properties

The diagram that contributes to the thermodynamic potential is given by [Yao and Zhang, PRA 2018]

$$\Omega_{\rm int} = \sum_{m,\mathbf{q}} \int_{-\infty}^{\infty} \frac{dz}{\pi} \frac{1}{e^{\beta z}-1} {\rm Im}\{\ln[1 + g_m^2 \Pi_m(\mathbf{q}, z+i0^+) G_0^M(\mathbf{q}, z+i0^+)]\}$$



Take into account the free thermodynamic potential from both molecule and fermion, one finds finally

$$\Omega = \Omega_0^F - \sum_{m,\mathbf{q}} \int \frac{dz}{\pi} \frac{1}{e^{\beta z}-1} \delta_m(\mathbf{q}, z)$$

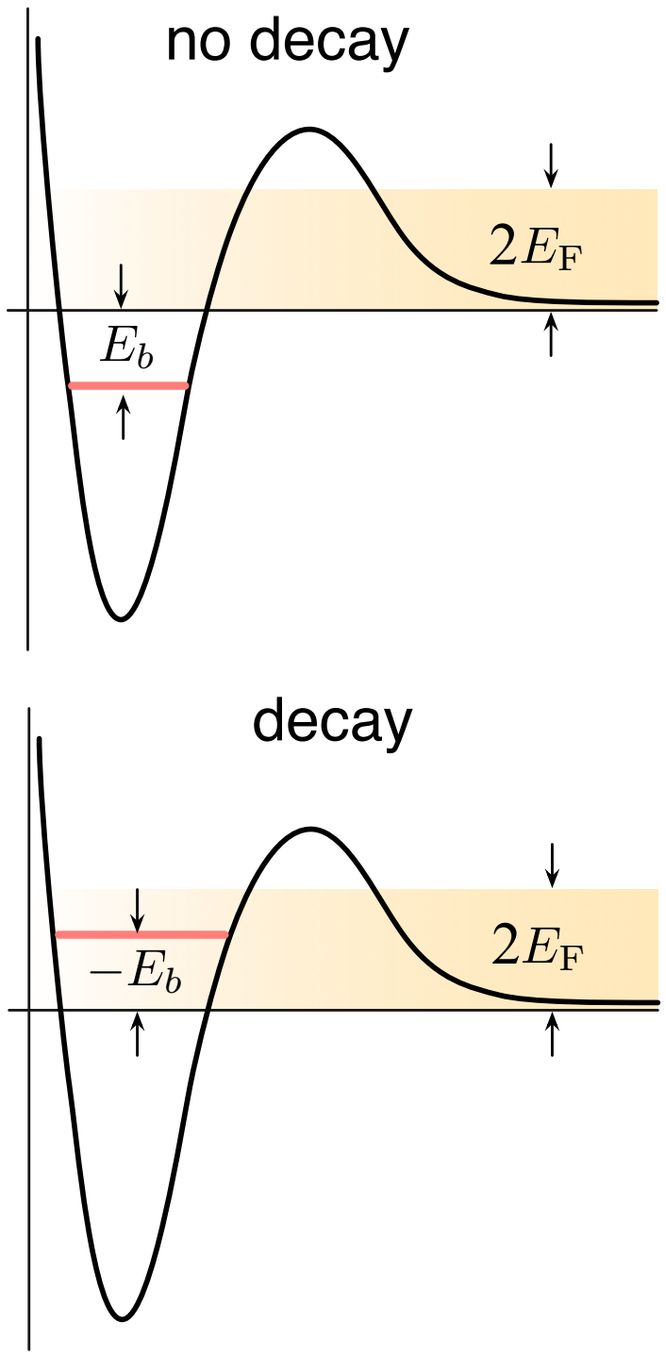
This is identical to the expression obtained for a single channel model, provided that one identify $\delta_m(\mathbf{q}, z) = -\arg[\Gamma_m^{-1}(\mathbf{q}, z)]$.

$$\Gamma_m^{-1}(\mathbf{q}, z) = \frac{m^2 r_m}{4\pi} \bar{z} + \frac{m}{4\pi v_m} + \Pi_m^r(\mathbf{q}, z)$$

$$\Pi_m^r(\mathbf{q}, z) = \Pi_m(\mathbf{q}, z) - \frac{m}{\Omega} \sum_{\mathbf{k}} 1 - \frac{m\bar{z}}{2\Omega} \sum_{\mathbf{k}} \frac{1}{\epsilon_{\mathbf{k}}}$$

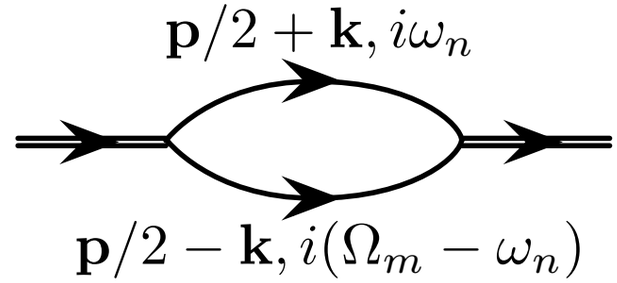
$$\bar{z} = z - \epsilon_{\mathbf{q}/2} + 2\mu$$

Decay of Quasi-bound Dimer: two-body



To calculate the life-time (decay rate) of the dimer, we compute the simplest dimer self energy

$$\Sigma_b(i\Omega_m, \mathbf{p}) =$$



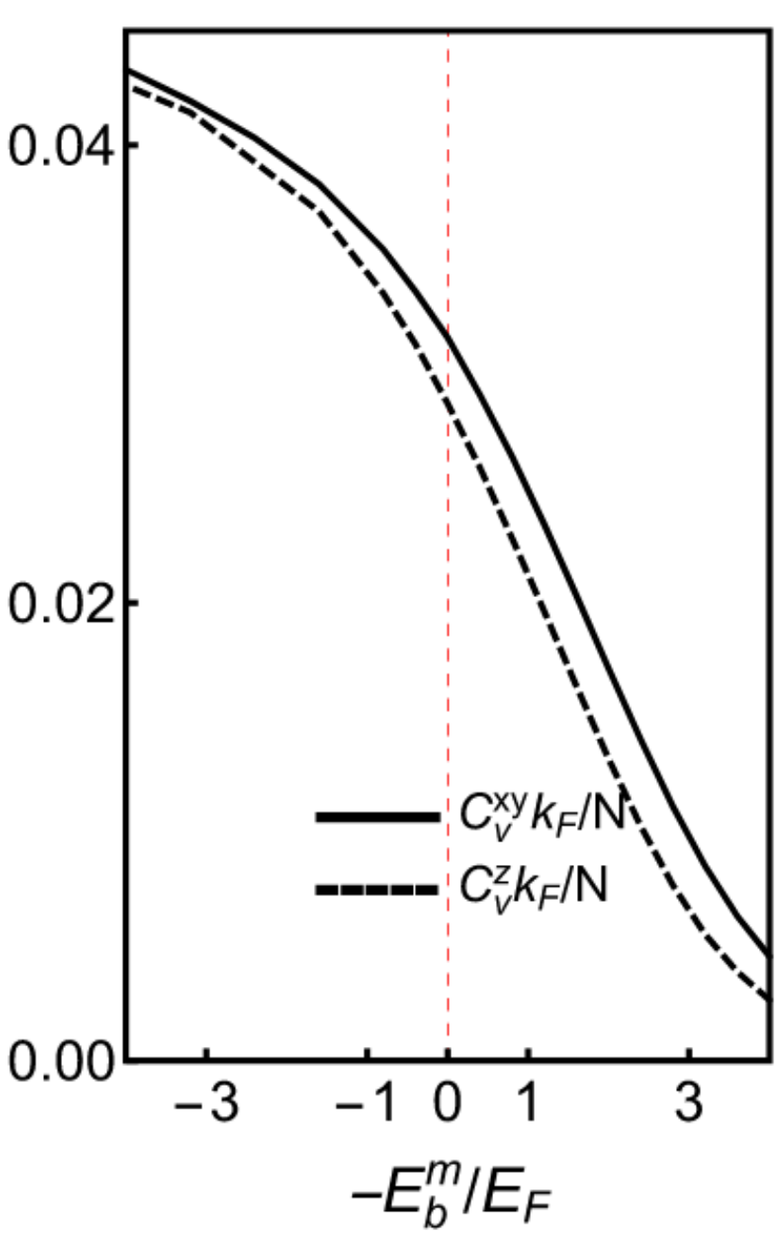
For scattering in vacuum, we find that for a shallow dimer at rest $p = 0$, the decay rate is given by

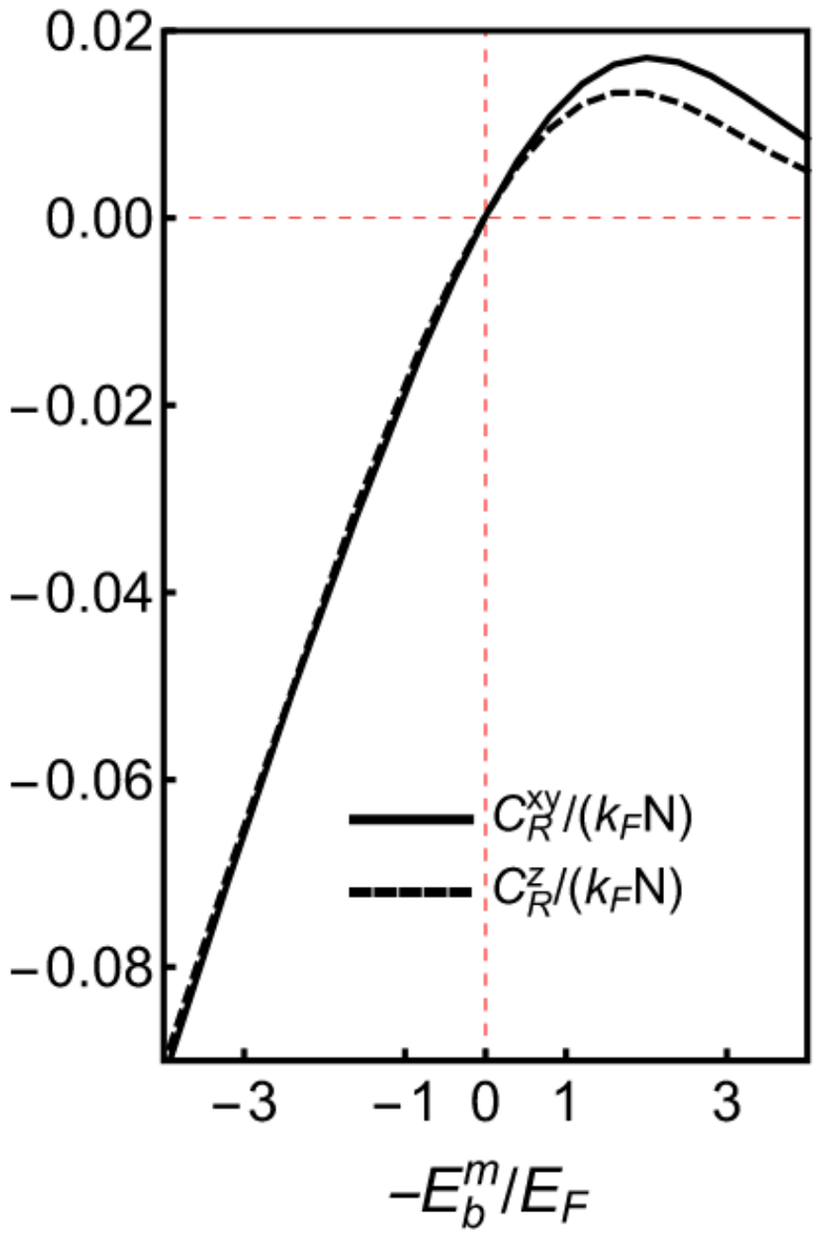
$$\hbar\gamma_m = \frac{\hbar^2}{m r_m}\left(-\frac{m E_b^m}{\hbar^2}\right)^{3/2} \sim \epsilon_{\rm F}(k_F/r_1)(E_b^m/\epsilon_F)^{3/2}$$

Close to resonance, this is quite small. Thus, only a narrow window in energy in the scattering states are affected by the presence of quasi-bound state.

Normal state properties

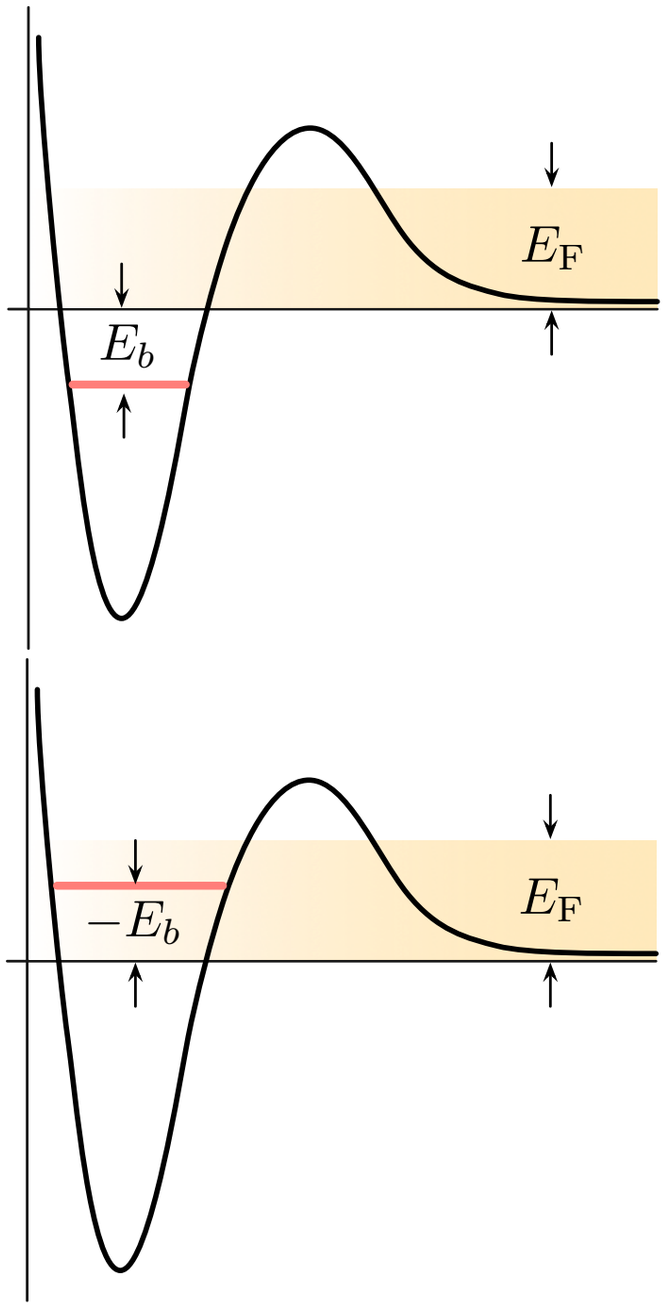
Several thermodynamic quantities can be computed from the NSR scheme, for example, free energy, chemical potential and contact parameters. $k_F/r_m \approx 0.04$.





Normal state properties

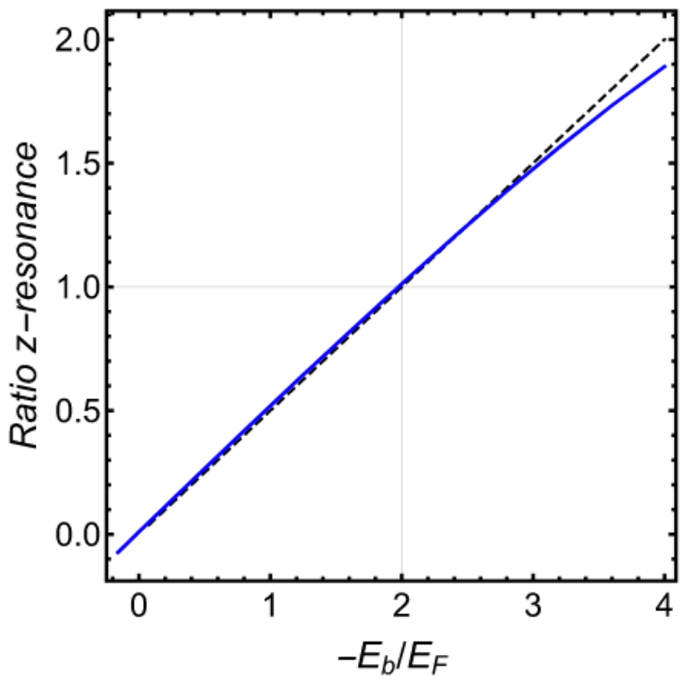
In fact, a simple model of boson-fermion mixture can explain most of findings by NSR.

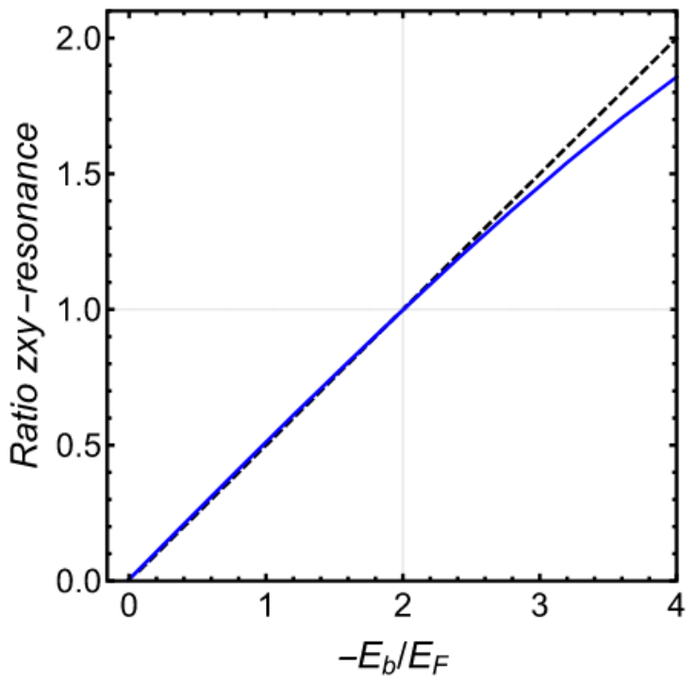


When the bound state is below the continuum, all fermions form molecules with bound state energy $E_b = \hbar^2/(mvr_1)$

When the quasi-bound state is above the continuum, all fermions with energy above E form bound states. Otherwise there is no interactions. So contacts come entirely from bound states.

$$\frac{C_{R,m}/k_F}{C_{v,m}k_F} = -\frac{1}{2}\frac{E_b^m}{E_F}$$





Conclusion

Introduction to p-wave system and its comparison with its s-wave counterpart;

1. Energy (momentum)-dependent scattering;
2. Effective range cannot be neglected and cannot be tuned to zero;

Low-energy scattering and universal relations;

1. Similar to the s-wave case but now requires extra parameters;
2. Sub-leading term can be important;
3. Generalisation to cases when rotation symmetry is broken (explicitly or spontaneously)

Normal state and Fermi liquid properties;

1. Narrow resonance leads to simple picture of molecule-fermion mixture;
2. Effective range makes contributions to quantum fluctuations;
3. Some consequence for transport.

Three-body recombination.

1. Effective range changes the scaling law for three-body recombination;
2. Further studies needed to address the experimental observation.

Strongly Interacting Fermi Gases

Shizhong Zhang

The University of Hong Kong

Croucher Summer Course: Ultracold Atom Physics

May 24, 2024

Plan:

Lecture I:

Introduction to p-wave system and its comparison with its s-wave counterpart;
Low-energy scattering and universal relations;
Normal state and Fermi liquid properties;
Three-body recombination.

Lecture II:

Bulk viscosity and effects of effective range; connection to SO(2,1) symmetry;
Viscosity sum rule and high frequency tails;
Viscosity in 1D p-wave Fermi system;
Emergent s-wave scattering in 1D p-wave Fermi system.

Recap: Low-energy scattering and unitary limit

For s-wave scattering:

$$f_s(k) = \frac{1}{-1/a_s - r_e k^2 - ik}$$

Unitary limit corresponds to $a_s = \pm\infty$ and $r_e = 0$, $f_s(k) = i/k$. This corresponds to a scale invariant interaction. Finite a_s or r_e will lead to the breaking of scale invariance.

For p-wave scattering:

$$f_p(k) = \frac{k^2 \cos\chi}{-1/v - r_1 k^2 - ik^3}$$

Unitary limit corresponds to $v = \pm\infty$ and $r_1 = 0$, $f_s(k) = i\cos\chi/k$. This also corresponds to a scale invariant interaction. Finite v or r_1 will lead to the breaking of scale invariance.

In fact, there is a dynamical SO(2,1) symmetry that governs the TDSE, and this leads to the so-called conformal tower states [cf. Lecture by Qi Zhou].

Scale invariant limit

Typically, one can incorporate the scattering via the Bethe-Peierls boundary condition (without effective range correction). In the s-wave case:

$$\lim_{|\vec{x}_i-\vec{y}_j|\to 0} \Psi(\{\vec{x}_i\},\{\vec{y}_j\}) = \left\{ \frac{1}{|\vec{x}_i-\vec{y}_j|} - \frac{1}{a_s} \right\} A(\{\vec{x}'_i\},\{\vec{y}'_j\})$$

If we perform a scaling transformation:

$$\vec{x}_i \to \vec{x}_i/\lambda(t);\ \text{and}\ \vec{y}_j \to \vec{y}_j/\lambda(t)$$

The Bethe-Peierls boundary condition becomes

$$\lim_{|\vec{x}_i-\vec{y}_j|\to 0} \Psi(\{\vec{x}_i/\lambda\},\{\vec{y}_j/\lambda\}) = \left\{ \frac{\lambda}{|\vec{x}_i-\vec{y}_j|} - \frac{1}{a_s} \right\} A(\{\vec{x}'_i/\lambda\},\{\vec{y}'_j/\lambda\})$$

$$= \left\{ \frac{1}{|\vec{x}_i-\vec{y}_j|} - \frac{1}{\lambda a_s} \right\} A'(\{\vec{x}'_i/\lambda\},\{\vec{y}'_j/\lambda\})$$

There are two limits for which the boundary condition is invariant: (1) non-interacting limit when $a_s = 0$ and (2) unitary limit when $a_s = \pm\infty$.

In the presence of effective range, scaling symmetry is broken.

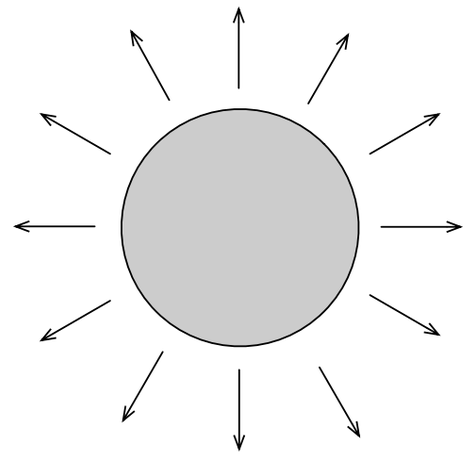
Scale invariant limit

In fact, it is possible to show that in a harmonic trap, the TDSE

$$i\hbar\frac{\partial\Psi(\vec{X},t)}{\partial t} = \left[-\frac{\hbar^2}{2m}\nabla^2_{\vec{X}} + \frac{1}{2}m\omega(t)^2\vec{X}^2\right]\Psi(\vec{X},t)$$

is exactly satisfied with

$$\Psi(\vec{X},t) = \frac{e^{i\theta(t)}}{\lambda(t)^{3N/2}}\exp\left[\frac{im\dot{\lambda}(t)}{2\hbar\lambda(t)}\vec{X}^2\right]\Psi(\vec{X}/\lambda(t),t)$$



$$\vec{v} = \lambda(t)(x,y,z)$$

Provided that

$$\ddot{\lambda}(t) + \omega^2(t)\lambda(t) - \frac{\omega^2(t)}{\lambda^3(t)} = 0$$

For small oscillations $\lambda(t) = 1 + \epsilon(t), \quad \epsilon(t) \ll 1$

$$\ddot{\epsilon}(t) + 4\omega^2\epsilon(t) = 0$$

Namely, the oscillation is exactly at twice the harmonic trap frequency with no damping. Bulk viscosity vanishes at all frequencies.

See, for example, Werner and Castin, in The BCS-BEC Crossover and the Unitary Fermi Gas, ed. Wilhelm Zwerger (Springer, 2011). E. Taylor and M. Randeria, PRA **81** 053601 (2010)

Viscosity

When internal friction is taken into account, the Euler equation becomes

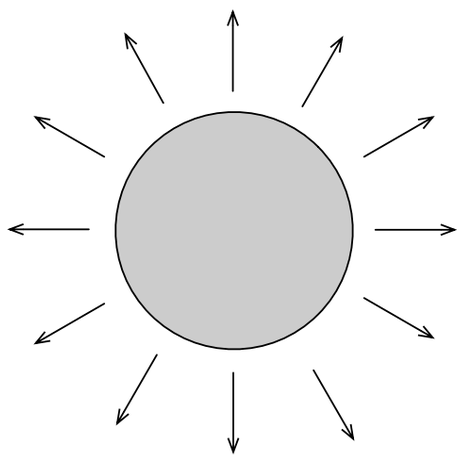
$$\frac{\partial}{\partial t}(\rho v_i) = -\frac{\partial \Pi_{ik}}{\partial x_k}$$

The momentum flux density operator Π_{ik} takes the form:

$$\Pi_{ik} = p\delta_{ik} + \rho v_i v_k - \eta\left(\frac{\partial v_i}{\partial x_k} + \frac{\partial v_k}{\partial x_i} - \frac{2}{3}\delta_{ik}\frac{\partial v_l}{\partial x_l}\right) + \zeta\delta_{ik}\frac{\partial v_l}{\partial x_l}$$

η is the shear viscosity and ζ is the bulk viscosity.

In one dimension, there is no shear motion, so only bulk viscosity remains.



$$\vec{v} = \lambda(t)(x, y, z)$$

When system has scaling symmetry (when scattering length diverges + zero effective range), bulk viscosity should vanish.

This happens in the non-interacting and unitary limit.

Kubo formula for bulk and shear viscosities

To calculate the viscosities, we use Kubo formula [Luttinger 1964]

Bulk: $$\zeta(\omega) = \frac{\mathrm{Im}\chi_R(\omega)}{\omega} \qquad \chi_{\mathrm{R}}(\omega) = i\int_0^\infty dt e^{-i\omega t}\left\langle[\widehat{\Pi}(t), \widehat{\Pi}(0)]\right\rangle$$

Shear: $$\eta(\omega) = \frac{\mathrm{Im}\chi_R^{(xy)}(\omega)}{\omega} \qquad \chi_{\mathrm{R}}^{(xy)}(\omega) = i\int_0^\infty e^{-i\omega t}\left\langle[\widehat{\Pi}^{(xy)}(t), \widehat{\Pi}^{(xy)}(0)]\right\rangle$$

In the case of bulk viscosity, the correlation can be written in terms of contact correlation functions [$\widehat{C}_{a_\ell} = -\partial\widehat{H}/\partial a_\ell^{-1}$ and $\widehat{C}_{R_\ell}(t) = -\partial\widehat{H}/\partial R_\ell$]

$$\widehat{\Pi}(t) = 2\widehat{H}_\ell + \frac{(2\ell+1)\widehat{C}_{a_\ell}(t)}{a_\ell} + (2\ell-1)\widehat{C}_{R_\ell}(t)R_\ell$$

While for shear viscosity, it is given by

$$\widehat{\Pi}^{(xy)} = \int d\mathbf{r}\psi_\sigma^\dagger(\mathbf{r})(-i\boldsymbol{\nabla}_x)(-i\boldsymbol{\nabla}_y)\psi_\sigma(\mathbf{r})$$

Calculations on s-wave, see Nishida (2019), Enss (2019) and Hoffman (2019). Results for p-wave are given by Maki and Zhang (2020).

An Explicit Calculation in 3D

A two-channel model (for both s-wave and p-wave):

$$\widehat{H}_\ell = \sum_{\mathbf{k},\sigma}\left(\frac{k^2}{2}-\mu\right)\psi^\dagger_\sigma(\mathbf{k})\psi_\sigma(\mathbf{k}) + \sum_{\mathbf{Q}}\sum_{m=-\ell}^{\ell}\left(\frac{Q^2}{4}+\nu-2\mu\right)d^\dagger_m(\mathbf{Q})d_m(\mathbf{Q})$$

$$+ g\sum_{m=-\ell}^{\ell}\sum_{\mathbf{Q},\mathbf{k},\sigma}\sqrt{\frac{4\pi}{V}}k^\ell\left[Y_{\ell,m}(\hat{\mathbf{k}})d^\dagger_m(\mathbf{Q})\psi_\sigma\left(\frac{\mathbf{Q}}{2}+\mathbf{k}\right)\psi_{\bar{\sigma}}\left(\frac{\mathbf{Q}}{2}-\mathbf{k}\right)+h.c.\right].$$

s-wave: $\sigma=-\bar{\sigma}=1/2;\ m=0$ p-wave: $\sigma=\bar{\sigma};\ m=-1,0,1$

The coupling constants g and ν can be related to (a_s, r_e) and (v, r_1) via standard renormalisation for two-particle scattering, and will be denoted as (a_ℓ, R_ℓ) for both s- and p-wave cases.

Operator form for contacts (A_0=1, A_1=1/2): $\widehat{C}_{a_\ell} = -\partial\widehat{H}/\partial a_\ell^{-1}$ and $\widehat{C}_{R_\ell}(t) = -\partial\widehat{H}/\partial R_\ell$

$$\widehat{C}_{a_\ell}(t) = \sum_{m=-\ell}^{\ell}\frac{g^2}{4\pi A_\ell}\int d\mathbf{R}d^\dagger_m(\mathbf{R},t)d_m(\mathbf{R},t)$$

$$\widehat{C}_{R_\ell}(t) = \sum_{m=-\ell}^{\ell}\frac{g^2}{4\pi A_\ell}\int d\mathbf{R}d^\dagger_m(\mathbf{R},t)\left(i\partial_t+\frac{\nabla^2}{4}\right)d_m(\mathbf{R},t)$$

Bulk viscosity

In equilibrium, any correlation with Hamiltonian is zero, so we are left with only correlators between various contacts [Maki and Zhang, PRL (2020)]:

$$\chi_{\rm R}(\omega) = i\int_0^\infty dt e^{-i\omega t}\left\langle\left[\frac{(2\ell+1)\widehat{C}_{a_\ell}(t)}{a_\ell} + (2\ell-1)\widehat{C}_{R_\ell}(t)R_\ell, \frac{(2\ell+1)\widehat{C}_{a_\ell}(0)}{a_\ell} + (2\ell-1)\widehat{C}_{R_\ell}(0)R_\ell\right]\right\rangle$$

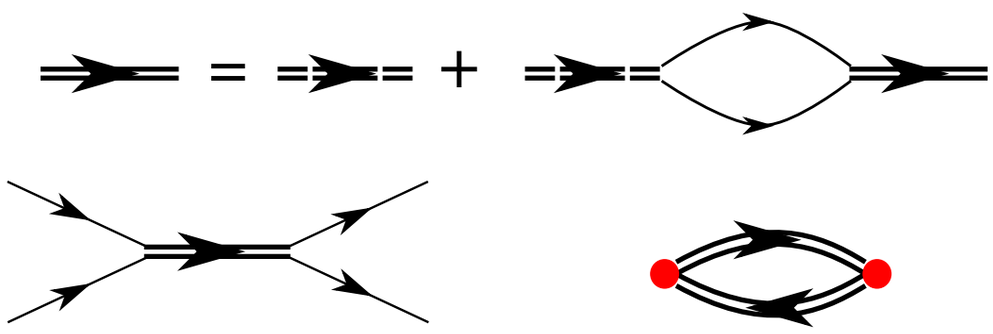
$$\zeta_\ell(\omega) = \zeta_{a,a}(\omega) + 2\zeta_{a,R}(\omega) + \zeta_{R,R}(\omega)$$

Thus, the calculation of bulk viscosity is reduced to the computation of four-dimer correlation function. For example:

$$\zeta_{a,R}(\omega) = \frac{(2\ell-1)(2\ell+1)R_\ell}{9a_\ell V}\sum_{m,\mathbf{Q}}\int_{-\infty}^{\infty}\frac{dx}{\pi}$$

$$\times{\rm Im}\left[\frac{g^2 D(\mathbf{Q}, x+i\delta)}{4\pi A_l}\right]{\rm Im}\left[\frac{g^2 D(\mathbf{Q}, x+\omega+i\delta)}{4\pi A_l}\right]$$

$$\times f(x,\omega)\left(x - \frac{Q^2}{4} + 2\mu + \frac{\omega}{2}\right)$$



$$f(x,\omega) = \omega^{-1}[n_B(x) - n_B(x+\omega)]$$

Bulk viscosity at high temperature

In the high temperature limit, analytic results can be obtained. Here high-temperature limit is defined by

s-wave case: $$a_s^{-2} \ll k_B T, \omega \ll r_e^{-2}$$

p-wave case: $$1/(v r_1) \ll k_B T, \omega \ll r_1^2$$

	High frequency limit (leading term)	zero-frequency limit $(a_s = \pm\infty; v = \pm\infty)$
s-wave	$\frac{\sqrt{\omega} r_e^2}{36V}\left[C_{a_s}\left(1 - \frac{1}{a_s^2\omega} - \frac{4}{a_s r_e \omega}\right) + \frac{C_{r_e}}{\omega}\right]$	$\frac{2^{5/2} z^2}{9}\frac{r_e^2}{\lambda_T^5}$
p-wave	$\frac{\sqrt{\omega}}{36V}\left[C_{a_1}\left(1 + \frac{10}{v r_1 \omega}\right) + \frac{7 C_{r_1}}{2\omega}\right]$	$\frac{2^{5/2} z^2}{3}\frac{1}{r_1^2 \lambda_T^5}$

Note that at resonance for high frequency limit (very different for *s*- and *p*-wave):

$$\zeta_0(\omega, a_s \to \pm\infty) = \frac{r_e^2 C_{a_s}}{36V}\sqrt{\omega} \propto r_e^2$$

$$\zeta_1(\omega, v \to \pm\infty) = \frac{C_v}{36V}\sqrt{\omega} \propto r_1^{-1}$$

Bulk viscosity

To second order in fugacity, from the explicit expressions, it can be shown that

$$\zeta(\omega) = \zeta(-\omega)$$

A sum rule can also be derived for $\zeta(\omega)$

$$\int_{-\infty}^{\infty} \frac{d\omega}{\pi} \left(\zeta_\ell(\omega) - \zeta_\ell^d(\omega)\right) = -\frac{1}{9V}\left[(2\ell+1)^2 \frac{\partial C_{a_\ell}}{\partial a_\ell} + (2\ell-1)^2 \frac{\partial C_{R_\ell}}{\partial R_\ell^{-1}}\right.$$
$$\left.+(2\ell-1)(2\ell+1)\left(\frac{1}{a_\ell R_\ell}\frac{\partial C_{a_\ell}}{\partial R_\ell^{-1}} + a_\ell R_\ell \frac{\partial C_{R_\ell}}{\partial a_\ell}\right)\right]$$

In the s-wave case, this generalises the earlier calculation by Taylor and Randeria, PRA **81** 053610 (2010)

Shear viscosity at high temperature

In the high temperature limit, analytic results can be obtained. Here high-temperature limit is defined by

s-wave case: $$a_s^{-2} \ll k_B T, \omega \ll r_e^{-2}$$

p-wave case: $$1/(v r_1) \ll k_B T, \omega \ll r_1^2$$

	High frequency limit (leading term)	zero-frequency limit $(a_s = \pm\infty; v = \pm\infty)$
s-wave	$\frac{4}{15}\left(\frac{C_a}{V}\frac{1}{\sqrt{\omega}} + \frac{5C_{r_e}}{2V}\frac{1}{\omega^{3/2}}\right)$	$\frac{5\pi}{8\sqrt{2}\lambda_T^3}$
p-wave	$\frac{4}{15}\left(\frac{C_v}{V}\sqrt{\omega} + \frac{7C_r}{2V}\frac{1}{\sqrt{\omega}}\right)$	$\frac{5\pi}{48\times 2^{3/2}}\frac{r_1^2}{T\lambda_T^3} \propto r_1^2\sqrt{T}$

Note that at resonance for high frequency limit (very different for *s*- and *p*-wave):

$$\eta_0(\omega, a_s \to \infty) \propto \frac{1}{\sqrt{\omega}} \propto r_e^0$$

$$\eta_1(\omega, v \to \infty) \propto \sqrt{\omega} \propto r_1^{-1}$$

Viscosity in 1D p-wave Fermi gas

Now, let us consider the simple situation of 1D p-wave Fermi gas and investigate the bulk viscosity and its temperature dependence. We focus on the weak coupling limit for which the system is describable as a Luttinger liquid at low temperature.

$$H = \sum_k \xi_k \psi^\dagger(k)\psi(k) + g \sum_Q \sum_{k,\ell} \frac{k\ell}{4L} \psi^\dagger\left(\frac{Q}{2}+k\right)\psi^\dagger\left(\frac{Q}{2}-k\right)\psi\left(\frac{Q}{2}-\ell\right)\psi\left(\frac{Q}{2}+\ell\right)$$

The two-body T-matrix is

$$T^{-1}(Q,E) = \frac{1}{g} - \frac{m}{2}\int \frac{k^2 dk}{m(E-Q^2/4) - k^2 + i\delta} = \frac{m}{4a} - \frac{m}{4}\sqrt{-m(E-Q^2/4+i\delta)}$$

In the case of 1D p-wave, only the leading term in the effective range expansion is necessary, as in 3D s-wave case. In the high temperature and zero ω limit

	Weak coupling limit	Resonant limit
Bulk viscosity	$\zeta(\omega \to 0) \approx \frac{2}{\pi^{5/2}}(k_F a)^2 T^{1/2}$	$\zeta(\omega \to 0) \approx \frac{k_F^2}{T^{5/2}a^2} \ln\left(\frac{4}{a^2\omega e^{-\gamma_E}}\right)$

Maki and Zhang, 2023, see also Tanaka and Nishida (2022)

Viscosity in 1D p-wave Fermi gas

Instead of the usual canonical form of the Luttinger liquid, it turned out to be more appropriate for our purpose to use the dressed chiral modes: $H = H_{LL} + V$

$$H_{LL} = \frac{v}{2}\int dx[(\partial x\varphi_L(x))^2 + (\partial x\varphi_R(x))^2]$$

$$V = \eta_- \int dx[(\partial_x\varphi_L(x))^3 - (\partial_x\varphi_R(x))^3] + \eta_+ \int dx[(\partial_x\varphi_L(x))^2\partial_x\varphi_R(x) - (\partial_x\varphi_R(x))^2\partial_x\varphi_L(x)]$$

In this representation, η_-−term has the same structure as that arises from the quadratic band correction, but now includes interaction effects.

$$\eta_- = \sqrt{\frac{2\pi}{K}}\frac{1}{24m}\left[3 + K^2 + \frac{3mk_F}{2\pi}g(K^2-1)\right] \quad \to \frac{\sqrt{2\pi}}{6}\frac{1}{m} \qquad K = \sqrt{\frac{1 - mgk_F/(2\pi)}{1 + mgk_F/(2\pi)}}$$

Since this term only reproduces the non-interacting band structure, we do not expect it to give rise to a non-zero bulk viscosity. We shall focus on the η_+−term.

$$\eta_+ = \sqrt{\frac{2\pi}{K}}\frac{1}{8m}\left[1 - K^2 - \frac{mk_F}{2\pi}g(1+3K^2)\right] \qquad \to \frac{\sqrt{2\pi}}{8m}\frac{mgk_F}{2\pi}$$

This term describes the genuine interaction effects that leads to a nonzero viscosity and we shall work to second order in g.

Viscosity in 1D p-wave Fermi gas

In terms of dressed chiral fields, relevant operators can be written as

$$\delta n(x) = -\sqrt{\frac{K}{2\pi}} \sum_r r \partial_x \varphi_r(x) \qquad v = v_F \sqrt{1 - \left(\frac{2k_F a}{\pi}\right)^2}$$

$$J(x) = -\sqrt{\frac{K}{2\pi}} \sum_r \left[v \partial_x \varphi_r(x) - (3\eta_- + \eta_+) r (\partial_x \varphi_r(x))^2 \right]$$

$$\Pi(x) = \sqrt{\frac{K}{2\pi}} \left[4(3\eta_- + \eta_+) H - 2v\eta_+ \sum_r \int dx \left(\partial_x \varphi_r(x) \partial_x \varphi_{\bar{r}}(x) - (\partial_x \varphi_r(x)^2) \right) \right]$$

One can then proceed with the computation of Kubo correlation function for bulk viscosity.

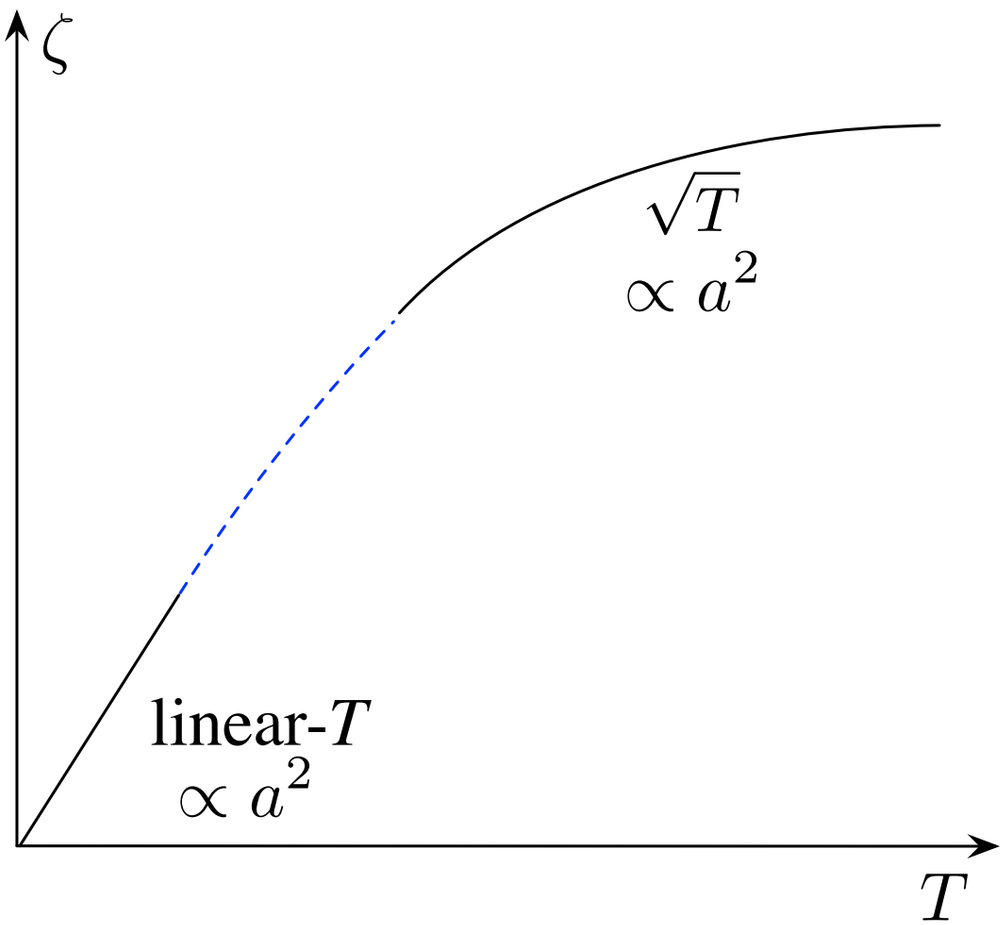
1. At thermal equilibrium, commutation with H is zero, so only η_+−term contributes; η_-−term does not contribute to bulk viscosity.

2. $\zeta \propto \eta_+^2$

Viscosity in 1D p-wave Fermi gas

Focusing on the η_+-term, we can calculate its contribution to viscosity using Kubo formula [Maki and Zhang, PRA (2023)]

$$\zeta(\omega) = \frac{K\eta_+^2}{4\pi v}\frac{\omega}{\tanh(\beta\omega/4)} = \frac{mk_F a^2}{72\pi}\frac{\omega}{\tanh(\beta\omega/4)}$$



Drude weight

What we have calculated so far is the regular part of the viscosity spectral function, but there could also be a Drude weight

$$\zeta(\omega) = D\delta(\omega) + \zeta_{\text{reg}}(\omega)$$

In the perturbative calculation at high temperature, one find that (Nishida 2023)

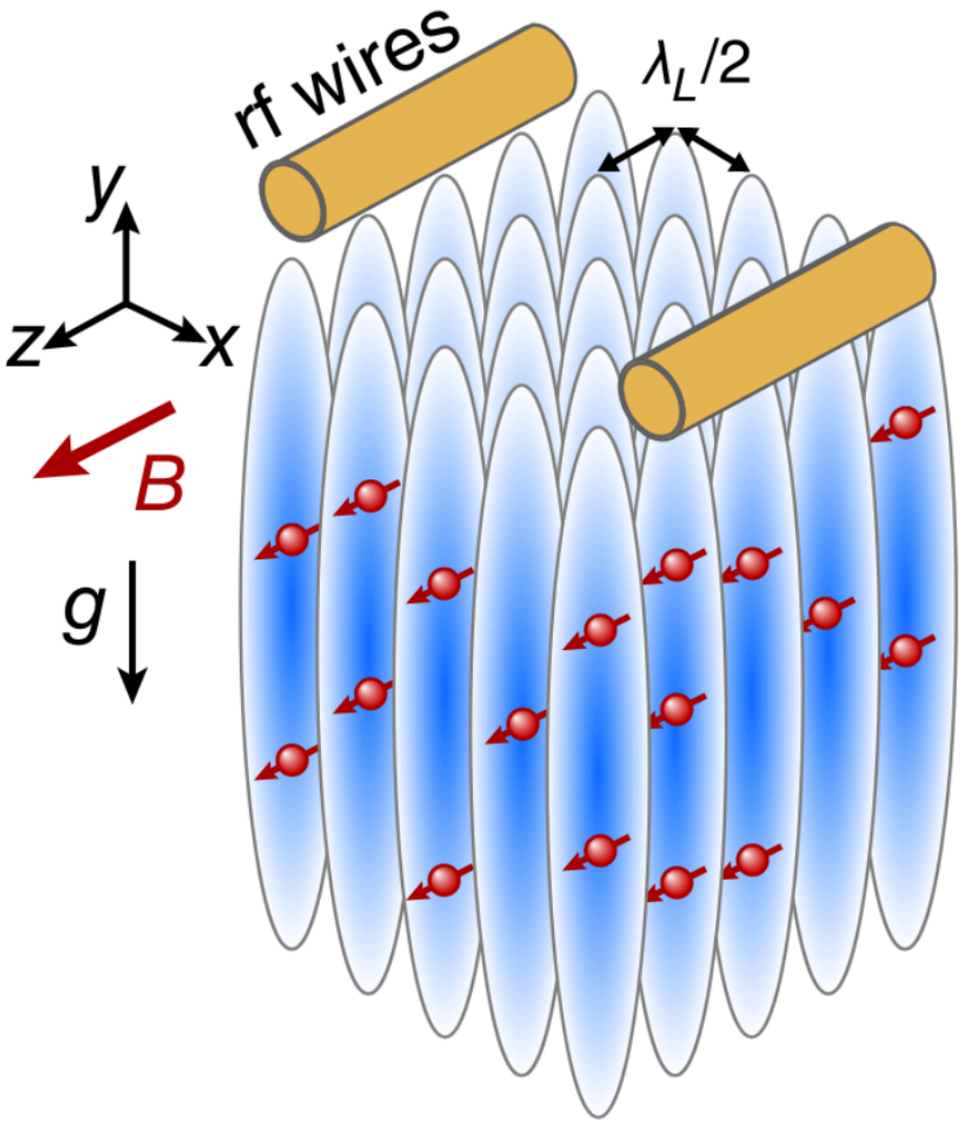
$$D \sim a^6 \quad a \to 0$$

$$D \sim 1/a^4 \quad a \to \infty$$

Now, in fact the 1D p-wave Fermi model without effective range is dual to Lieb-Liniger model and can be solved exactly. It will be desirable to calculate the Drude weight D using Bethe ansatz… Work in progress!

Experimental system

When confined into an one-dimensional tube, a single component Fermi gas can exhibit s-wave scattering when transverse collision channels are available.



Kenneth Jackson et al, PRX **13** 021013 (2023)

Single particle Hamiltonian:

$$H_0 = \frac{1}{2m}\mathbf{p}^2 + \frac{1}{2}m(\omega_x^2 x^2 + \omega_y^2 y^2 + \omega_z^2 z^2)$$

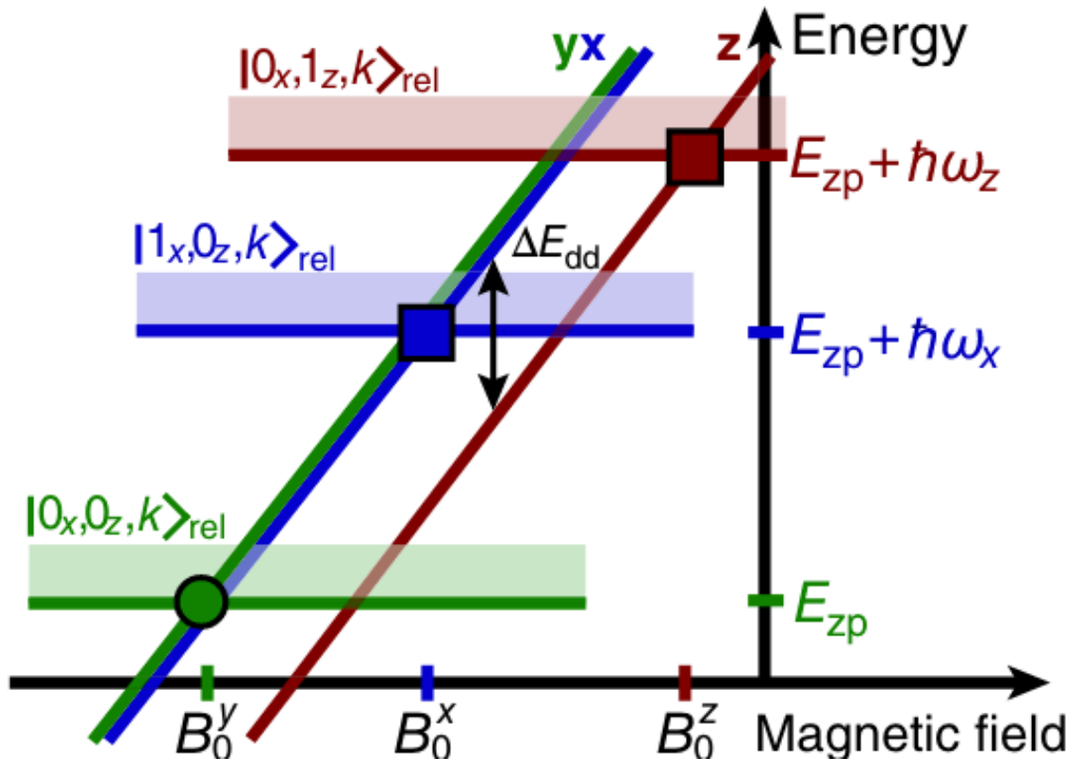
With simple replacement of the effective mass, this is also the non-interacting Hamiltonian for the relative motion of two particles in the quasi-1D trap.

Assuming confinement along $y-$direction is weak and setting $\omega_y = 0$. Then the relative eigenstates are labeled as $|n_x, n_z, k\rangle$.

Continuum	Asymptotic state	Transverse parity	q1D parity	Resonant dimer
Zeroth	$\|0_x, 0_z, k\rangle_{\text{rel}}$	Even	Odd	y dimer
First	$\|1_x, 0_z, k\rangle_{\text{rel}}$	Odd	Even	x dimer
Second	$\|0_x, 1_z, k\rangle_{\text{rel}}$	Odd	Even	z dimer

Experimental system

One complication for ^{40}K is that there is a dipole splitting of the resonance, in this case, since the magnetic field is along z, the $xy-$resonance (corresponds to $m = \pm 1$) split from the $z-$resonance ($m = 0$) by about $0.5G$.



Continuum	Asymptotic state	Transverse parity	q1D parity	Resonant dimer
Zeroth	$\|0_x, 0_z, k\rangle_{\text{rel}}$	Even	Odd	y dimer
First	$\|1_x, 0_z, k\rangle_{\text{rel}}$	Odd	Even	x dimer
Second	$\|0_x, 1_z, k\rangle_{\text{rel}}$	Odd	Even	z dimer

Note:

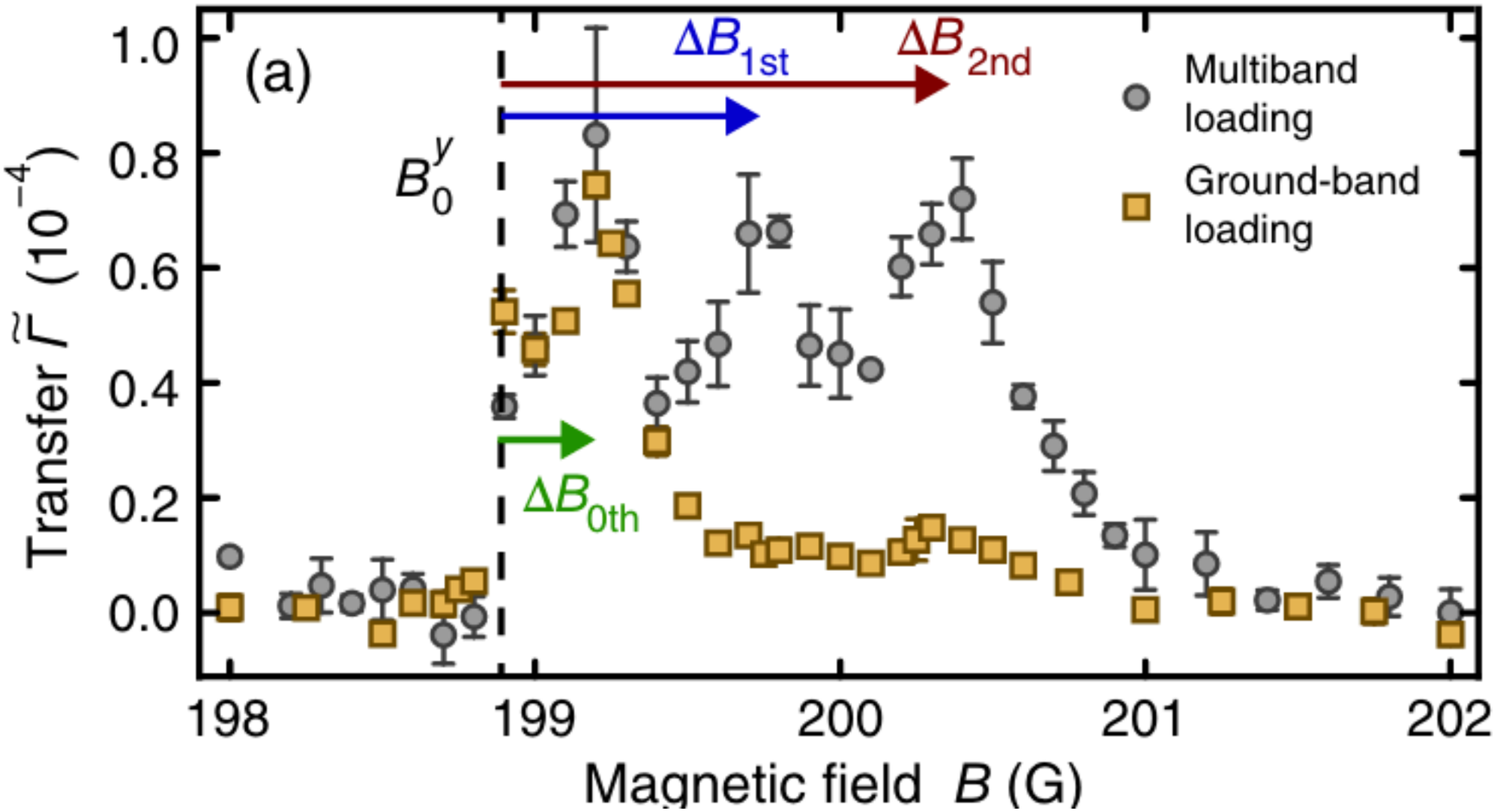
1. Excitation along $x, z-$direction means that the energy is shifted by $\hbar\omega_{x,z}$.
2. Dipole interaction shift the $z-$resonance up in magnetic field.
3. P-wave dimer energy changes approximately linearly with magnetic field.

$$E_b = -\frac{\hbar^2}{mvr_1} \propto (B - B_0)$$

4. The hybridisation of p-wave dimer and scattering states needs to respect symmetry considerations (Franck-Condon Principle).

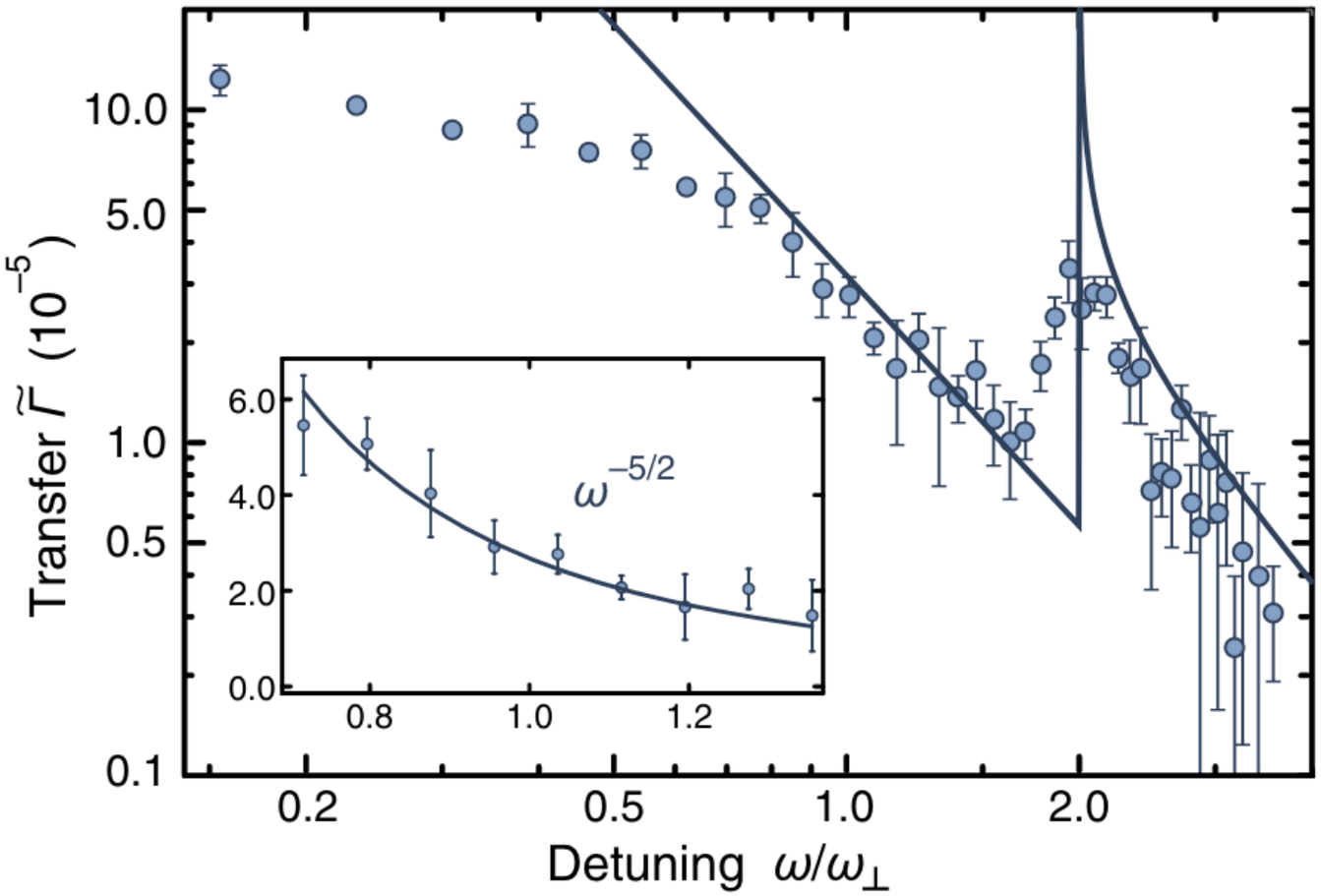
Experimental system

We thus expect three resonances with first resonance of p-wave character in quasi-1D direction and the following two resonances of s-wave character. The later two resonances only occur when there are occupations of the relevant excited band.



Confirmation of s-wave channel

So, what are the consequences of the new modality of scattering? This turned out to be quite spectacular for radio-frequency response. Using OPE (operator product expansion):



s-wave RF tail:

$$\Gamma^{(\pm 1)}(\omega) = \frac{\Omega^2 C_e^{(\pm 1)}}{2}\left[\frac{\theta(\omega)}{\omega^{5/2}} + \frac{2\theta(\omega - 2\omega_\perp)}{(\omega - 2\omega_\perp)^{1/2}\omega^2}\right]$$

p-wave RF tail:

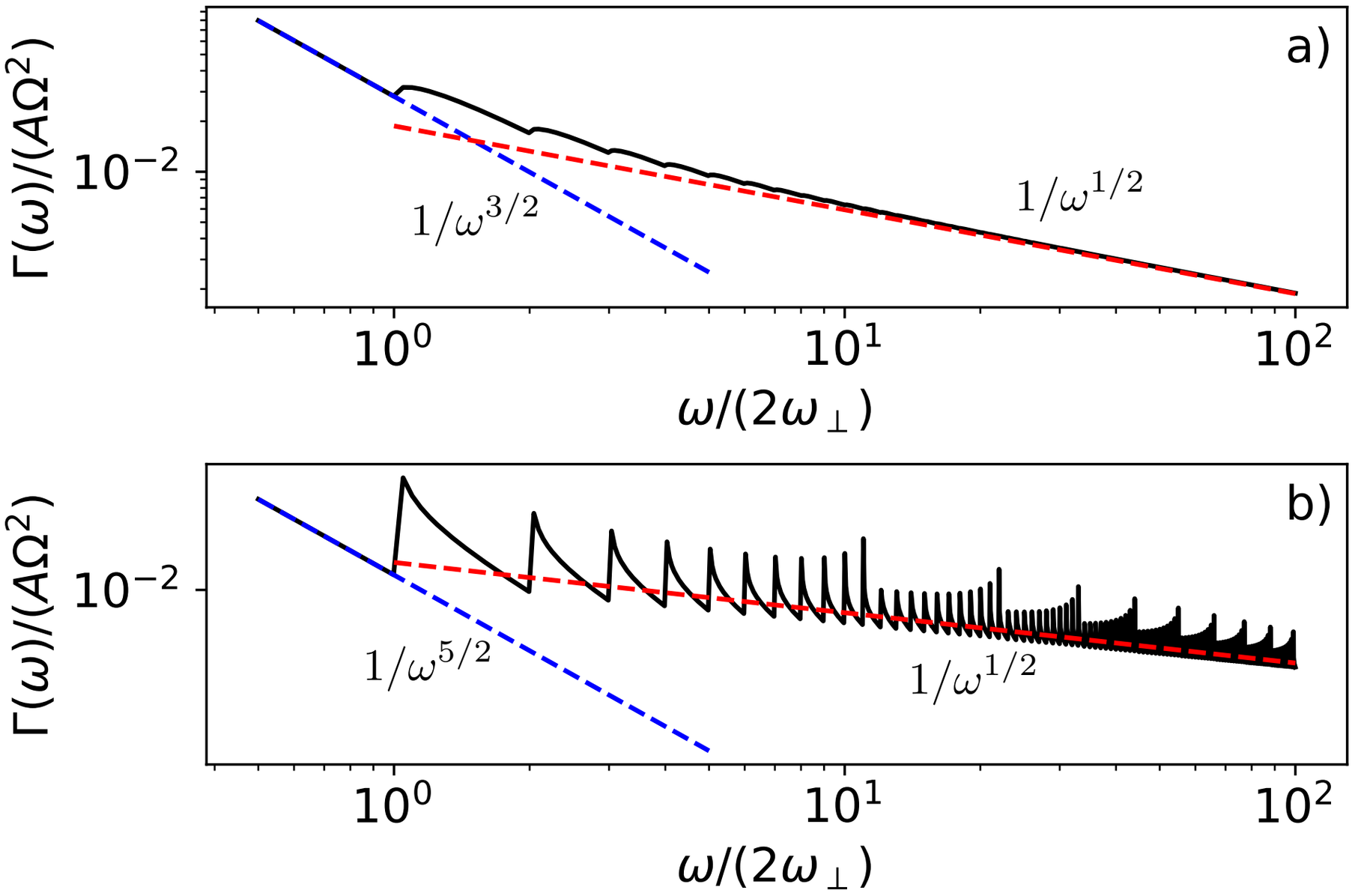
$$\Gamma^{(0)}(\omega) = \frac{\Omega^2 C_o}{2}\left[\frac{\theta(\omega)}{\omega^{3/2}} + \frac{\theta(\omega - 2\omega_\perp)\sqrt{\omega - 2\omega_\perp}}{\omega^2}\right]$$

Intuitive Fermi Golden rule analysis.

In fact, at still higher frequencies, similar behaviour repeats and one expects to the response should crossover to 3D. Indeed this is the case.

Crossover in RF

In the asymptotic frequency regime, there is a crossover from 1D to 3D behaviour with unique features appearing when a new scattering channel becomes available. Similar crossover also occurs in 2D-3D, and has been seen experimentally at Toronto.



Conclusion

Introduction to p-wave system and its comparison with its s-wave counterpart;

1. Energy (momentum)-dependent scattering;
2. Effective range cannot be neglected and cannot be tuned to zero;

Low-energy scattering and universal relations;

1. Similar to the s-wave case but now requires extra parameters;
2. Sub-leading term can be important;
3. Generalisation to cases when rotation symmetry is broken (explicitly or spontaneously)

Normal state and Fermi liquid properties;

1. Narrow resonance leads to simple picture of molecule-fermion mixture;

Viscosity;

1. Singular dependences on the effective range, opposite to that in s-wave case
2. 1D viscosity non-analytic in ω, hydrodynamic longtime tails…

Confinement in 1D:

1. New modality in 1D scattering
2. Crossover from 1D to 3D can be explored in RF.